OPTIMALLY STABLE HIGH-ORDER DISCONTINUOUS GALERKIN SCHEMES FOR MULTIDIMENSIONAL LINEAR HYPERBOLIC SYSTEMS

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Kinetic transport equations are widely used in many applications of computational physics, including thermal radiative transfer, neutron transport, and semiconductor modeling. These model equations describe the time evolution of a particle distribution in phase space. A standard form of the linear kinetic transport equation is the following integro-differential equation:

$$\frac{\partial f}{\partial t} + \mathbf{\Omega} \cdot \nabla f = -\sigma_t f + \frac{\sigma_s}{4\pi} \int_{S^2} f(t,x,\mathbf{\Omega}) \, d\Omega,$$

where $f(t,x,\mathbf{\Omega}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}_{\geq 0}$ is the particle distribution function, $t \in \mathbb{R}_{\geq 0}$ is time, $x \in \mathbb{R}^3$ is the spatial position, $\mathbf{\Omega} \in \mathbb{R}^3$ is the particle propagation velocity ($\|\mathbf{\Omega}\|_2 = 1$), and $\sigma_t$ and $\sigma_s$ are the total and scattering cross-sections, respectively.

Standard techniques such as discrete ordinates ($S_N$), spherical harmonics ($P_N$), and hybrid discrete ($H^T_N$), approximate the distribution function, $f(t,x,\mathbf{\Omega})$, via some ansatz of the form:

$$f(t,x,\mathbf{\Omega}) \approx \sum_{\ell} q_{\ell}(t,x) \varphi_{\ell}(\mathbf{\Omega}),$$

where $\varphi_{\ell}(\mathbf{\Omega}) : \mathbb{R}^3 \mapsto \mathbb{R}_{\geq 0}$ is some appropriate basis in the velocity coordinate (e.g., $S_N$: delta functions, $P_N$: spherical harmonics, $H^T_N$: piecewise polynomials). Plugging this ansatz into (1) yields to a linear system of partial differential equations of the form:

$$q_{,t} + A^x q_{,x} + A^y q_{,y} + A^z q_{,z} = C q,$$

for some constant-coefficient matrices $A^x, A^y, A^z$, with the property that the following matrix:

$$A(n) = n^x A^x + n^y A^y + n^z A^z,$$

is diagonalizable with only real eigenvalues for all $\|n\| = 1$ (i.e., linear system (3) is hyperbolic).

In this work we develop a novel discontinuous Galerkin method for solving linear hyperbolic systems of hyperbolic of the form (3). The proposed method is a genuinely multidimensional extension of the classical Lax-Wendroff DG method. The main advantage of the modified method is that it has a significantly larger region of stability and has improved accuracy. We provide a rigorous von Neumann stability analysis for the proposed scheme and test it on several test cases, including examples relevant to the radiative transfer equation (1).