OPTIMALLY STABLE HIGH-ORDER DISCONTINUOUS GALERKIN SCHEMES FOR MULTIDIMENSIONAL LINEAR HYPERBOLIC SYSTEMS

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Kinetic transport equations are widely used in many applications of computational physics, including thermal radiative transfer, neutron transport, and semiconductor modeling. These model equations describe the time evolution of a particle distribution in phase space. A standard form of the linear kinetic transport equation is the following integro-differential equation:

$$\frac{\partial f}{\partial t} + \underline{\Omega} \cdot \underline{\nabla} f = -\sigma_t f + \frac{\sigma_s}{4\pi} \iint_{\mathbb{S}^2} f\left(t, \underline{x}, \underline{\Omega}\right) \mathrm{d}\underline{\Omega}, \tag{1}$$

where $f\left(t,\underline{x},\underline{\Omega}\right) : \mathbb{R}_{\ge 0} \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}_{\ge 0}$ is the particle distribution function, $t \in \mathbb{R}_{\ge 0}$ is time, $\underline{x} \in \mathbb{R}^3$ is the spatial position, $\underline{\Omega} \in \mathbb{R}^3$ is the particle propagation velocity ($\|\underline{\Omega}\|_2 = 1$), and σ_t and σ_s are the total and scattering cross-sections, respectively.

Standard techniques such as discrete ordinates (S_N), spherical harmonics (P_N), and hybrid discrete (H_N^T), approximate the distribution function, $f\left(t,\underline{x},\underline{\Omega}\right)$, via some ansatz of the form:

$$f\left(t,\underline{x},\underline{\Omega}\right) \approx \sum_{\ell} q_\ell\left(t,\underline{x}\right)\, \varphi_\ell\left(\underline{\Omega}\right), \tag{2}$$

where $\varphi_\ell\left(\underline{\Omega}\right) : \mathbb{R}^3 \mapsto \mathbb{R}_{\ge 0}$ is some appropriate basis in the velocity coordinate (e.g., S_N: delta functions, P_N: spherical harmonics, H_N^T: piecewise polynomials). Plugging this ansatz into (1) yields to a linear system of partial differential equations of the form:

$$\underline{q}_{,t} + \underline{\underline{A}}^x \underline{q}_{,x} + \underline{\underline{A}}^y \underline{q}_{,y} + \underline{\underline{A}}^z \underline{q}_{,z} = \underline{\underline{C}}\, \underline{q}, \tag{3}$$

for some constant-coefficient matrices $\underline{\underline{A}}^x$, $\underline{\underline{A}}^y$, and $\underline{\underline{A}}^z$ with the property that the following matrix:

$$\underline{\underline{A}}\left(\underline{n}\right) = n^x\, \underline{\underline{A}}^x + n^y\, \underline{\underline{A}}^y + n^z\, \underline{\underline{A}}^z, \tag{4}$$

is diagonalizable with only real eigenvalues for all $\|\underline{n}\| = 1$ (i.e., linear system (3) is hyperbolic).

In this work we develop a novel discontinuous Galerkin method for solving linear hyperbolic systems of hyperbolic of the form (3). The proposed method is a genuinely multidimensional extension of the classical Lax-Wendroff DG method. The main advantage of the modified method is that it has a significantly larger region of stability and has improved accuracy. We provide a rigorous von Neumann stability analysis for the proposed scheme and test it on several test cases, including examples relevant to the radiative transfer equation (1).