DGSEM Approximation of a Well-Posed Overset Grid Formulation for Hyperbolic Systems

David A. Kopriva*
Department of Mathematics, The Florida State University, Tallahassee, FL 32306, USA
and
Computational Science Research Center, San Diego State University, San Diego, CA, USA.

Jan Nordström†
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 524 Auckland Park 2006, South Africa

Gregor J. Gassner‡
Department for Mathematics and Computer Science; Center for Data and Simulation Science, University of Cologne, Weyertal 86-90, 50931, Cologne, Germany

Overset grid methods have long been used to simplify the application of numerical methods to complex geometric configurations. They approximate the solutions of PDEs by splitting a domain into multiple overlapping subdomains and approximating the equations on each subdomain with coupling conditions to keep them consistent with the original single domain problem. The expectation is that the approximate solutions on the overlapping subdomains converge to the solution of the original problem on the full domain.

Stability of the coupling procedures has been a practical and theoretical issue with overset mesh methods [1], and fully multidimensional stability proofs are not available. One of the issues with finding a general stability proof is that the original initial-boundary value problem (IBVP) on which the approximation is based needs to be well-posed in the first place. It turns out that in multiple space dimensions, commonly used characteristic one-way coupling conditions do not lead to a well-posed overset domain problem, making it impossible to construct a convergent numerical approximation. To remedy that, we recently developed [2] a well-posed and conservative formulation for linear hyperbolic systems that uses penalty terms and *two-way* coupling to ensure that the overset problem is energy bounded and has a solution identical to the original. The formulation is flexible in that it allows one to weight the contributions of the multiple solutions in the overlap regions to reflect, for instance, the fact that one approximation may be more accurate than another due to different resolutions.

In this work, we study approximate solutions of the novel formulation using a discontinuous Galerkin spectral element approximation (DGSEM) [3],[4]. As an example, we show here an approximation of a solution to the wave equation

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}_x = 0 \quad x \in \Omega = [0, 7] \tag{1}$$

on overlapping domains Ω_u and Ω_v, as sketched in Fig. 1, with $a = 0$, $b = 2.5$, $c = 5.2$ and $d = 7$. Solutions at $t = 24$ are compared to the exact in Fig. 2. For this smooth solution, Fig. 2 shows that convergence on both grids is exponential, as desired.

In the presentation we will show how to apply the DGSEM to the novel overset domain problem, and present further numerical results assessing the weighting of the domains and the effect that additional coupling in the interior of the overlap domain has.

*Professor Emeritus, Department of Mathematics, Florida State University and San Diego State University, `kopriva@math.fsu.edu`

†Professor, Department of Mathematics, Computational Mathematics, Linköping University, SE-581 83 Linköping, Sweden, and Distinguished Visiting Professor at Department of Mathematics and Applied Mathematics, University of Johannesburg, P.O. Box 524 Auckland Park 2006, South Africa, `jan.nordstrom@liu.se`

‡Professor, Department for Mathematics and Computer Science; Center for Data and Simulation Science, University of Cologne, Weyertal 86-90, 50931, Cologne, Germany, `ggassner@uni-koeln.de`

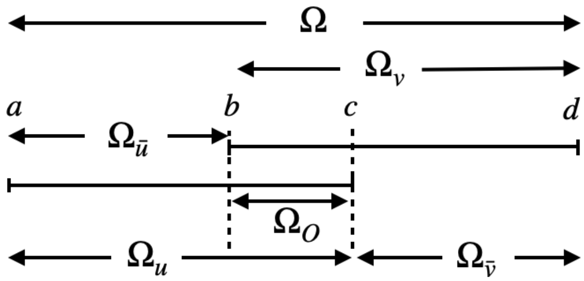

Fig. 1   Overset domain definitions in 1D

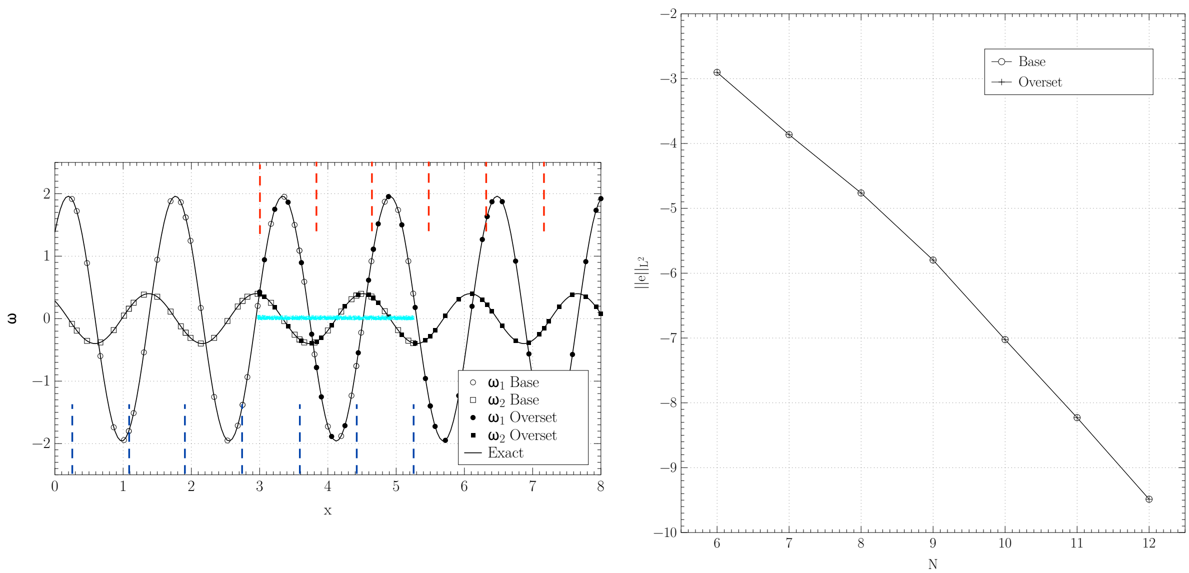

Fig. 2   Left: Exact and computed solutions for a sinusoidal problem. Vertical lines mark the element boundaries. The horizontal line marks the overlap region. Right: Convergence of the L^2 error as a function of polynomial order.

References

[1] Sharan, N., Pantano, C., and Bodony, D. J., "Time-stable overset grid method for hyperbolic problems using summation-by-parts operators," Journal of Computational Physics, Vol. 361, 2018, pp. 199–230. doi:https://doi.org/10.1016/j.jcp.2018.01.049, URL `https://www.sciencedirect.com/science/article/pii/S0021999118300597`.

[2] Kopriva, D. A., Nordström, J., and Gassner, G. J., "On the theoretical foundation of overset grid methods for hyperbolic problems: Well-posedness and conservation," Journal of Computational Physics, Vol. 448, 2022, p. 110732. doi:https://doi.org/10.1016/j.jcp.2021.110732, URL `https://www.sciencedirect.com/science/article/pii/S0021999121006276`.

[3] Black, K., "A conservative spectral element method for the approximation of compressible fluid flow," KYBERNETIKA, Vol. 35, No. 1, 1999, pp. 133–146.

[4] Kopriva, D. A., Implementing Spectral Methods for Partial Differential Equations, Scientific Computation, Springer, 2009.